Istituzioni di Matematiche
CdL Scienze Biologiche

Esercizio I

Determinare inf e sup

$$X = \left\{ \log_{\frac{1}{2}}(2n^2-3n+2) \ : \ \begin{array}{l} n\in\mathbb{N} \\ n\geq 1 \end{array} \right\}$$

precisando se si tratta di min/max.

Modo 1 (classico) $\qquad \log_{\frac{1}{2}} z \geq 0 \iff 0 < z \leq 1$

$2n^2-3n+2 \geq 1$?

$2n^2-3n+1 \geq 0 \qquad \Delta = 9-8 = 1 \qquad , \ \cancel{n \leq \frac{3-1}{4}} \cup n \geq \frac{3+1}{4}$

$\cancel{n \leq \frac{1}{2}} \cup n \geq 1$

$\Rightarrow$ se $a_n = \log_{\frac{1}{2}}(2n^2-3n+2)$

si ha $\qquad a_n < 0 \quad \forall n \geq 1$

$\Rightarrow$ Gli elementi di X sono formati da numeri negativi

$\Rightarrow \quad 0 \in X^*$

Cerco maggiorante $\quad x<0$

$$\log_{\frac{1}{2}}(2n^2-3n+2) \leq x \qquad \forall n \geq 1$$

$$2n^2-3n+2 \geq \left(\tfrac{1}{2}\right)^x$$

Impongo $\forall n \in \mathbb{N}$ $n \geq 1$

$$\boxed{2n^2-3n+\left(2-\left(\tfrac{1}{2}\right)^x\right) \geq 0}$$

Ciò accade $\iff$ $\Delta \leq 0$

$$\Delta = 9 - 8\left(2 - \frac{1}{2^x}\right) \leq 0$$

$$= 9 - 16 + \frac{2^3}{2^x} \leq 0$$

$$\iff 2^{3-x} \leq 7$$

$$\iff \log_2 2^{3-x} \leq \log_2 7$$

$$\iff 3-x \leq \log_2 7$$

$$\iff x \geq \underbrace{3 - \log_2 7}_{>0}$$

Mai

$\Rightarrow$ Maggioranti $x<0$ non ne abbiamo!

$\Rightarrow \sup X = 0$

Verifico se
$$0 = \log_{\frac{1}{2}}(2n^2-3n+2)$$

$$1 = 2n^2-3n+2$$

$$2n^2-3n+1 = 0 \qquad n=1 \quad \text{OK}$$

$$\Rightarrow \boxed{\max X = 0}$$

Cerco minoranti $y<0$

$$y \le \log_{\frac{1}{2}}(2n^2-3n+2) \quad \forall n \in \mathbb{N}_{n\geq 1}$$

$$\frac{1}{2^y} \geq 2n^2-3n+2$$

$$\Leftrightarrow \quad 2n^2-3n+\left(2-\frac{1}{2^y}\right) \le 0 \quad \forall n\in\mathbb{N} \quad n\geq 1$$

Se $\Delta>0$ $\quad \frac{3-\sqrt{\Delta}}{4} \le n \le \frac{3+\sqrt{\Delta}}{4}$

non è possibile $\forall n \in \mathbb{N}$

$\Delta \le 0$ NO

$\Rightarrow$ X non è limitato inferiorm.

$$\inf X = -\infty$$

Modo 2 (Alternativo) uso le funzioni

$$a_n = \log_{\frac{1}{2}}(2n^2-3n+2)$$

$$f(x) = \log_{\frac{1}{2}}(2x^2-3x+2) \longleftrightarrow \boxed{x \geq 1}$$

$$f'(x) = \underbrace{\frac{1}{2x^2-3x+2}}_{>0} \cdot \frac{4x-3}{\underbrace{\log \frac{1}{2}}_{<0}}$$

$$f'(x) \geq 0 \iff \boxed{\begin{array}{ll} 2x^2-3x+2 \geq 0 & \text{sempre} \\ \Delta = 9-16<0 & \end{array}}$$

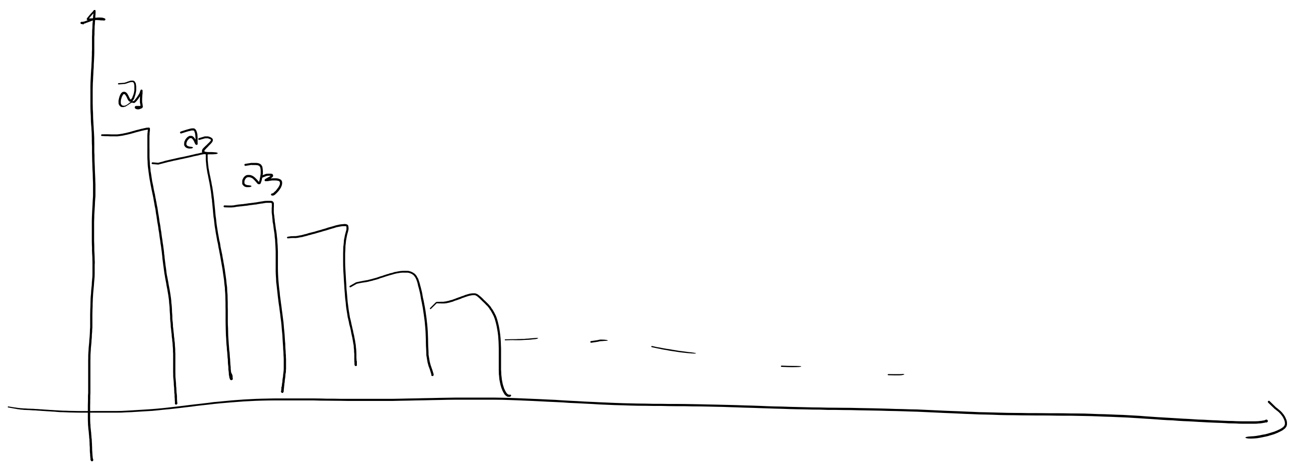
$$\Updownarrow$$

$$4x-3 \leq 0 \iff \boxed{x \leq \frac{3}{4}}$$

$\Longrightarrow$ $f(x)$ è decrescente in $[1, +\infty[$

$\Longrightarrow$ $a_n = f(n) \quad \forall n \in \mathbb{N}$

$\Longrightarrow$ $(a_n)_n$ è decrescente



$$\Longrightarrow \quad \max X = a_1 = \log_{\frac{1}{2}}(2-3+2) = 0$$

$$\inf X = \lim_{n\to+\infty} \log_{\frac{1}{2}} \underbrace{(2n^2-3n+2)}_{+\infty} = -\infty$$

Esercizio II

$$\lim_{x \to \frac{3}{2}} \frac{9-4x^2}{(5^{2x-3}-1)} \cdot \frac{1}{\operatorname{sen}\left(\frac{\pi}{2x+1}\right)} =$$

$$= \left(\frac{9-4\cdot\frac{9}{4}}{5^{2\cdot\frac{3}{2}-3}-1} \cdot \frac{1}{\operatorname{sen}\frac{\pi}{4}} \right)$$

$$= \left(\frac{0}{0} \cdot \frac{1}{\operatorname{sen}\frac{\pi}{4}} \right)$$

f.i. $\neq 0$

A parte

$$\lim_{x \to \frac{3}{2}} \frac{9-4x^2}{5^{2x-3}-1} = (*)$$

<u>modo 1</u> (classico (lim not))

Ricordo $\lim_{z \to 0} \frac{a^z-1}{z} = \lg a$

$$(*) = \lim_{x \to \frac{3}{2}} \frac{2x-3}{5^{2x-3}-1} \cdot \frac{9-4x^2}{2x-3}$$

$z = 2x-3 \xrightarrow{x \to \frac{3}{2}} 0$

$$\boxed{\lim_{z \to 0} \frac{z}{5^z-1} = \frac{1}{\lg 5}}$$

$$\lim_{x \to \frac{3}{2}} \frac{(\cancel{3-2x})(3+2x)}{\cancel{2x-3}}$$

$$= \lim -(3+2x)$$

$\lim_{x \to \frac{3}{2}}$

$= -\left(3 + 2 \cdot \frac{3}{2}\right)$

$= -6$

Conclusione $\displaystyle\lim_{x \to \frac{3}{2}} \frac{9 - 4x^2}{(5^{2x-3} - 1)\left(\sin \frac{\pi}{2x+1}\right)} =$

$$= \frac{1}{\lg 5} \cdot (-6) \cdot \frac{1}{\sin \frac{\pi}{4}}$$

Modo 2 (De l'Hopital)

A parte $\displaystyle\lim_{x \to \frac{3}{2}} \frac{9 - 4x^2}{5^{2x-3} - 1} \quad \left(\frac{0}{0} \text{ f.i.}\right)$

$$\overset{\text{De l'Hop}}{=} \lim_{x \to \frac{3}{2}} \frac{-8x}{5^{2x-3} \cdot \lg 5 \cdot (2)} =$$

$$= \frac{-\cancel{8}^{\,\cancel{4}\,2} \cdot \frac{3}{\cancel{2}\,1}}{1 \cdot \lg 5 \cdot \cancel{2}} =$$

$$= \frac{1}{\lg 5} (-6)$$

Conclusione $\displaystyle\lim_{x \to \frac{3}{2}} \frac{9 - 4x^2}{5^{2x-3} - 1} \frac{1}{\sin\left(\frac{\pi}{2x+1}\right)} = \frac{1}{\lg 5} (-6) \cdot \frac{1}{\sin \frac{\pi}{4}}$

Attenzione allora

$$\left(5^{2x-3}-1\right)' = \left(5^{2x-3}\right)' =$$

$$= \left(5^{z}\right)' = 5^{z}\cdot \lg 5 \cdot z'$$

$z = 2x-3$

$$= 5^{2x-3}\cdot \lg 5 \cdot (2)$$

Esercizio III

Assegnata la Funzione

$$f(x) = \begin{cases} x^3 - 4x^2 - 5x + 1 & \text{se } x \in [-1, 0[\\ \lg(2x+1) & \text{se } x \in [0, +\infty[\end{cases}$$

$f: [-1, +\infty[\longrightarrow \mathbb{R}$

$$f'(x) = \begin{cases} 3x^2 - 8x - 5 & \text{se } x \in]-1, 0[\\ \frac{1}{2x+1}\cdot 2 & \text{se } x \in]0, +\infty[\end{cases}$$

Studio $f'(x) \geq 0$

Caso $x \in]-1, 0[$

$3x^2 - 8x - 5 \geq 0$

$$x \le \frac{8-2\sqrt{31}}{6} \cup x \ge \frac{8+2\sqrt{31}}{6}$$

$$\Delta = 8^2 - 4\cdot 3\cdot(-5) =$$

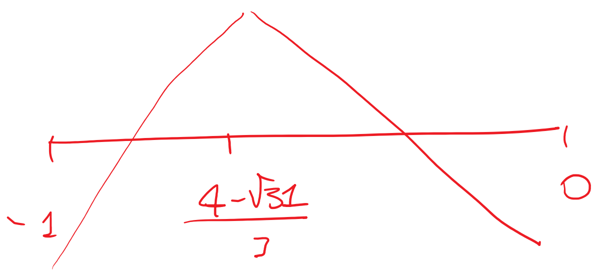
$$= 4^2\cdot 2^2 + 4\cdot 3\cdot 5 =$$

$$= 4\,(4\cdot 4 + 3\cdot 5)$$

$$= 4\,(31)$$

ossia

$$x \le \frac{4-\sqrt{31}}{3} \cup \cancel{x \ge \frac{4+\sqrt{31}}{3}}$$



$f(x)$ cresce in $\left[-1, \frac{4-\sqrt{31}}{3}\right]$

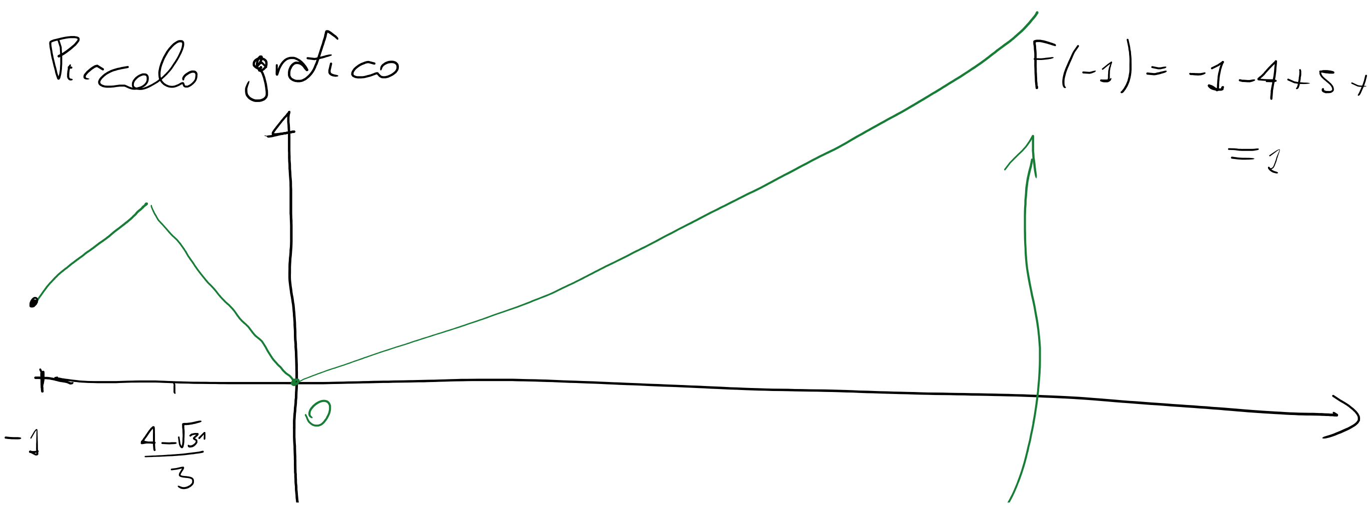
decresce in $\left[\frac{4-\sqrt{31}}{3}, 0\right]$

caso $x > 0$ $\quad f'(x) \ge 0$

$$\frac{2}{2x+1} \ge 0 \qquad \text{sempre}$$

$\Rightarrow f(x)$ cresce in $[0, +\infty[$

Piccolo grafico



$F(-1) = -1-4+5+1$

$= 1$

$-1 \qquad \frac{4-\sqrt{31}}{3} \qquad \Big| \, 0$

$$\lim_{x\to+\infty} \lg(2x+1) = +\infty$$

$\Longrightarrow \quad x = \frac{4-\sqrt{31}}{3}$ è max rel. (ma non assol.)

$x=-1$, $x=0$ minimi rel

$x=0$ min assoluto

<u>Riscrivo</u>

$$f'(x) = \begin{cases} 3x^2-8x-5 & x \in]-1,0[\\ \frac{2}{2x+1} & x \in]0,+\infty[\end{cases}$$

$$f''(x) = \begin{cases} 6x-8 & x \in]-1,0[\\ -\frac{4}{(2x+1)^2} & x \in]0,+\infty[\end{cases}$$

Ricordo $\boxed{\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{[f(x)]^2}}$

$$\left(\frac{2}{2x+1}\right)' = 2 \cdot \left(\frac{1}{2x+1}\right)' = -2 \, \frac{2}{(2x+1)^2}$$

$f''(x) \geq 0$ caso $x \in]-1,0[$

$6x-8 \geq 0 \iff 3x-4 \geq 0$

$\iff$

$$6x - 8 \geq 0 \iff 3x - 4 \geq 0$$
$$\iff \cancel{x \geq \frac{4}{3}}$$

$\Rightarrow f(x)$ è concava in $]-1, 0[$

caso $x \in]0, +\infty[$ $\qquad \dfrac{-4}{(2x+1)^2} \geq 0$ $\qquad$ mai

$\Rightarrow f(x)$ è concava in $]0, +\infty[$

$\Rightarrow$ No punti di flesso

Esercizio IV

Data la funzione $f(x) = \sqrt{|\lg x|}$

studiarne punti non derivab

CE $]0, +\infty[$

Ricordo $\lg x \geq 0 \iff x \geq 1$

$$f(x) = \begin{cases} \sqrt{\lg x} & \text{se } x \geq 1 \\ \sqrt{-\lg x} & \text{se } 0 < x < 1 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{1}{2\sqrt{\lg x}} \cdot \dfrac{1}{x} & \text{se } x > 1 \\ \dfrac{1}{2\sqrt{\ldots}} \cdot \left(-\dfrac{1}{x}\right) & \text{se } 0 < x < 1 \end{cases}$$

$$\left(\frac{1}{2\sqrt{-\lg x}} \cdot \left(-\frac{1}{x}\right) \quad \text{se } 0<x<1 \right.$$

Studio derivabilità in $x=1$

$$\lim_{x \to 1^-} f'(x) = \lim_{x \to 1^-} \frac{1}{2\underbrace{\sqrt{-\lg x}}_{\to 0^+}} \cdot \overbrace{\left(\frac{-1}{x}\right)}^{\to -1} = -\infty$$

$$\lim_{x \to 1^+} f'(x) = \lim_{x \to 1^+} \frac{1}{2\underbrace{\sqrt{\lg x}}_{\to 0^+}} \cdot \overbrace{\frac{1}{x}}^{\to 1} = +\infty$$

$\Longrightarrow$ $x=1$ è punto cuspidale

Esercizio Determinare inf e sup

$$A = \left\{ \left(\frac{1}{2}\right)^{n^2-3n+1} : n \in \mathbb{N} \right\}$$

Modo classico - A è formato da numeri positivi

Cerco minoranti $x>0$

$$\left(\frac{1}{2}\right)^{n^2-3n+1} \leq x \qquad \forall n \in \mathbb{N}$$

$$n^2-3n+1 \geq \lg_{\frac{1}{2}} x$$

Impongo

Sol

$$n^2-3n+(1-\log_{\frac{1}{2}}x) \geq 0 \qquad Sol = \forall n\in\mathbb{N}$$

$$\iff \Delta \leq 0$$

$$9-4(1-\log_{\frac{1}{2}}x) \leq 0$$

$$5+4\log_{\frac{1}{2}}x \leq 0$$

$$\log_{\frac{1}{2}}x \leq -\frac{5}{4}$$

$$x \geq \left(\frac{1}{2}\right)^{-\frac{5}{4}}$$

$$\Rightarrow \quad \sup A = \left(\frac{1}{2}\right)^{-\frac{5}{4}}$$

Verifica $\quad \left(\frac{1}{2}\right)^{-\frac{5}{4}} = \left(\frac{1}{2}\right)^{n^2-3n+1}$

$$\iff -\frac{5}{4} = n^2-3n+1$$

$$\iff n^2-3n+\frac{9}{4} = 0$$

$$\Delta = 9-4\cdot\frac{9}{4} = 0 \qquad n = \frac{3}{2} \quad \text{XD}$$

$$\Rightarrow \quad \sup A = \left(\frac{1}{2}\right)^{-\frac{5}{4}} \quad \text{ma non max}$$

Sappiamo che $0 \in A_*$ - Cerco minoranti

$y > 0$

$$y \le \left(\frac{1}{2}\right)^{n^2-3n+1}$$

$$\log_{\frac{1}{2}} y \ge n^2 - 3n + 1$$

$$n^2 - 3n + (1 - \log_{\frac{1}{2}} y) \le 0$$

$\forall n \in \mathbb{N}$

mai

$\Rightarrow$ non ci sono minoranti $y > 0$

$\Rightarrow \inf A = 0$

<u>Verifico</u> $0 = \left(\frac{1}{2}\right)^{n^2-3n+1}$ mai